On tree-valued Fleming-Viot processes

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Tree-valued Fleming-Viot processes model the evolving genealogical trees in a large population of constant size.

States: unlabeled infinite weighted trees.

Connections with the Donnelly-Kurtz lookdown approach, the Aldous CRT, the Bertoin-Le Gall flow of bridges, Cannings models, ...

In this talk: neutral, non-spatial setting.
The approach seem to works also in more general settings.

Tree-valued Moran process

Consider a Moran model of population size N in two-sided time.

The genealogical tree at any time t is described by a triple

$$[\![\{1, \ldots, N\}, \rho_t^N, \frac{1}{N}\sum_{i=1}^{N} \delta_i]\!] \, .$$

$\rho_t^N(i,j)$: genealogical distance between the individuals on lines i and j at time t
= twice the time difference to their most recent common ancestor

Metric measure spaces

- A metric measure space (X, ρ, μ) consists of
  - a complete and separable metric space (X, ρ)
  - a probability measure μ on $\mathcal{B}(X)$
- Let $x_1, x_2, \ldots$ be an μ-iid sequence. The distance matrix distribution of (X, ρ, μ) is the distribution of $(\rho(x_i, x_j))_{i,j \in \mathbb{N}}$.
- Metric measure spaces are called isomorphic iff they have the same distance matrix distribution.
- Metric measure spaces are said to converge in the Gromov-weak topology iff their distance matrix distributions converge weakly on $\mathcal{M}_1(\mathbb{R}^{\mathbb{N}^2})$.
- The space of isomorphy classes of metric measure spaces, endowed with the Gromov-weak topology, is Polish [Greven, Pfaffelhuber, Winter '09].

Metric measure spaces

A complete and separable metric on the space $\mathbb{M}$ of isomorphy classes of metric measure spaces is given by the Gromov-Prohorov distance

$$d_{\mathrm{GP}}((X,\rho,\mu),(\tilde{X},\tilde{\rho},\tilde{\mu})) = \inf d_{\mathrm{P}}^{Z}(\varphi(\mu),\tilde{\varphi}(\tilde{\mu}))$$

where the $\inf$ is over all metric spaces Z and isometric embeddings $\varphi : X \to Z$, $\tilde{\varphi} : \tilde{X} \to Z$.

Gromov reconstruction theorem (see [Vershik '02])
Metric measure spaces (X,ρ,μ), $(\tilde{X},\tilde{\rho},\tilde{\mu})$ are isomorphic iff there exists an isometry $\varphi : \mathrm{supp}\,\mu \to \mathrm{supp}\,\tilde{\mu}$ with $\tilde{\mu} = \varphi(\mu)$.

Tree-valued Fleming-Viot process

Recall the tree-valued Moran process
$\chi_t^N = [\![\{1, \dots, N\}, \rho_t^N, \frac{1}{N} \sum_{i=1}^N \delta_i]\!]$.

$(\chi_t^N)_{t \in \mathbb{R}}$ converges in distribution as $N \to \infty$ in the Skorohod metric on the space of càdlàg paths with values in $(\mathbb{M}, d_{\mathrm{GP}})$
limit: tree-valued Fleming-Viot process [Greven, Pfaffelhuber, Winter '13]

This approach corresponds to the classical construction of a measure-valued Fleming-Viot process.

[Depperschmidt, Greven, Pfaffelhuber '12]:
genetic types, mutation, selection, and recombination.

[Donnelly and Kurtz '96, '99]:
measure-valued processes from infinite particle systems (lookdown particle systems).

The Donnelly-Kurtz lookdown particle system

To ensure that every line is hit by arrows by a rate bounded in N, we let all arrows now point from lower to higher lines (now called ranks).

We obtain new genealogical distances $\rho_t^{N,\mathrm{ld}}(i,j)$, and we consider the tree-valued process

$$\chi_t^{N,\mathrm{ld}} := [\![\{1,\dots,N\}, \rho_t^{N,\mathrm{ld}}, \frac{1}{N}\sum_{i=1}^{N}\delta_i]\!], \quad t \in \mathbb{R}.$$

Fact

For each t, $\rho_t^{N,ld} \overset{d}{=} \rho_t^N$.
However, the processes $(\rho_t^{N,ld})_{t\in\mathbb{R}}$ and $(\rho_t^N)_{t\in\mathbb{R}}$ do not have the same law.
The processes $(\chi_t^N)_{t\in\mathbb{R}}$ and $(\chi_t^{N,ld})_{t\in\mathbb{R}}$ both have the law of a tree-valued Moran process.

The Donnelly-Kurtz lookdown particle system

A slight modification that does not change the previous fact:

Don’t kill the particle with the rank hit by the arrow.

Instead, increase its rank by one, and do the same for the particles from the higher rank onwards, except for the particle on the highest rank which is killed.

The Donnelly-Kurtz lookdown particle system

Now consider a particle system with infinitely many rank indexed by $\mathbb{N}$.

For each pair of ranks at rate 1, an arrow is shot from the lower to the higher rank.

When we can see just the particles on the first N ranks, we see a finite lookdown particle system.

TVFV in the lookdown particle system

Let $\rho_t(i,j)$ be the genealogical distance between the individuals with ranks i and j at time t.

Now we read off the state of a tree-valued Fleming-Viot process. Let (X_t, ρ_t) be the metric completion of $(\mathbb{N}, \rho_t)$.

A. s., the weak limit of probability measures on (X_t, ρ_t)

$$\mu_t := \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_i$$

exists.

On the space of isomorphy classes of metric measure spaces,

$$\lim_{N\to\infty} [\![\{1,\dots,N\}, \rho_t, \frac{1}{N}\sum_{i=1}^{N}\delta_i]\!] = [\![X_t, \rho_t, \mu_t]\!] \quad \text{a. s.}$$

Hence, $([\![X_t, \rho_t, \mu_t]\!])_{t\in\mathbb{R}}$ is a version of a tree-valued Fleming-Viot process.

The lookdown space

Consider the space $\mathbb{R} \times \mathbb{N}$ whose element (t, i) is the individual with rank i at time t.

Let $\rho((s, i), (t, j))$ be the genealogical distance between the individuals (s, i), (t, j). Let (Z, ρ) be the metric completion of $(\mathbb{R} \times \mathbb{N}, \rho)$. We call (Z, ρ) the lookdown space.

Equivalently, for each $t \in \mathbb{R}$,

$$\mu_t := \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{(t,i)}$$

exits on (Z, ρ), and the process $([\![Z, \rho, \mu_t]\!])_{t\in\mathbb{R}}$ is a version of a tree-valued Fleming-Viot process.

Construction of sample paths

Theorem (G. '17)

$$\lim_{n\to\infty} \sup_{t\in[-T,T]} d_P^Z(\mu_t, \frac{1}{N}\sum_{i=1}^{n} \delta_{(t,i)}) = 0 \quad a.\,s.$$

Thus:

$([\![Z,\rho,\mu_t]\!])_{t\in\mathbb{R}}$ gives a. s. the sample paths of a tree-valued Fleming-Viot process.

$\lim_{N\to\infty} [\![Z,\rho,\frac{1}{N}\sum_{i=1}^{N}\delta_{(t,i)}]\!] = [\![Z,\rho,\mu_t]\!]$ uniformly for t in compacts a. s.

Construction from a Brownian excursion

Let $(B_s)_{0 \leq s \leq \zeta}$ be a Brownian excursion obtained by conditioning Itô's measure to height larger than 1.

Let $(\mathcal{T}, d)$ be the CRT under the excursion.

Formally: $d(x, y) = B_x - 2 \min_{[x,y]} B + B_y$ for $0 \leq x \leq y \leq \zeta$, $\mathcal{T} = [0, \zeta]/d$

Construction from a Brownian excursion

We define normalized local time measures on $(\mathcal{T}, d)$:

Let $H = \max B$. For each $u \in (0, H)$ and $n \in \mathbb{N}$, let ℓ_u^n be the uniform measure on the roots of the n largest subtrees above u.

Fact

$\ell_u := w\text{-}\lim_{n\to\infty} \ell_u^n$ *exists for all* $u \in (0, H)$ *a. s.*

(This can be deduced from the uniform downcrossing representation for local times of [Chacon, Le Jan, Perkins, and Taylor '81].)

Construction from a Brownian excursion

Let L_u be the local time of B in height $u \in (0, H)$.

Define a time change $U(t), t \in \mathbb{R}$:

- $U(0) = 1$
- in height $u \in (0, H)$, travel with speed $4/L_u$.
- Let $U(t)$ be the height reached after traveling for time $|t|$

$$\begin{cases} \text{upwards:} & t > 0 \\ \text{downwards:} & t < 0. \end{cases}$$

Let $\mathcal{T}' = \mathcal{T} \setminus \{\text{root}, \text{tip}\}$.

Let d' be the metric on $\mathcal{T}'$ defined by the traveling time along the segments.

Theorem (G. '17)

$(\llbracket \mathcal{T}', d', \ell_{U(t)} \rrbracket)_{t \in \mathbb{R}}$ *is a tree-valued Fleming-Viot process.*

Construction from a Brownian excursion

Main part of proof: $(\mathcal{T}', d')$ is a lookdown space.

For $1 \leq i < j$ and $u \in (0, H)$, say j looks down on i at height u iff:
there is a branchpoint at height u that is the root of the i-th and j-th highest subtrees above u.

To show: $\left(\sum_{\substack{u \in (0,H): \\ j \text{ looks down on } i \text{ at } u}} \delta_{U^{-1}(u)} \right)_{1 \leq i < j}$ is a family of independent PP(1) processes on $\mathbb{R}$.

We can use the binary branching tree embedded in a Brownian excursion.
[J. and N. Berestycki '09] use excursion theory directly.

Path properties

- In the present setting:
  tvFV has a. s. continuous paths
  (see also [Greven, Pfaffelhuber, Winter ’13])
- For Ξ-lookdown:
  a. s. càdlàg paths
  jumps at multiple reproduction events
- In the Gromov-Hausdorff-Prohorov distance
  a. s. càdlàg paths
  jumps when parts of the genealogical tree break off

Further remarks

- The prelimiting tree-valued Moran processes can be replaced with tree-valued processes read off from Cannings models that satisfy the conditions of [Möhle and Sagitov '01].
- Work in progress with A. Blancas, S. Kliem, C. Tran, A. Wakolbinger:
  Instead of a Moran model, start out with a branching particle system with fecundity and viability selection

Summary

The Donnelly-Kurtz lookdown particle system gives a construction of the sample paths of tree-valued Fleming-Viot processes.

Central object is the random lookdown space which encodes all individuals ever alive and which is endowed with a probability measure for each time.

This construction is related to the Aldous continuum random tree and its local time measures.

Thank you!