The one-dimensional contact process and the KPP-equation with noise

Sandra Kliem

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The one-dimensional contact process

The nearest-neighbor contact process on $\mathbb{Z}$

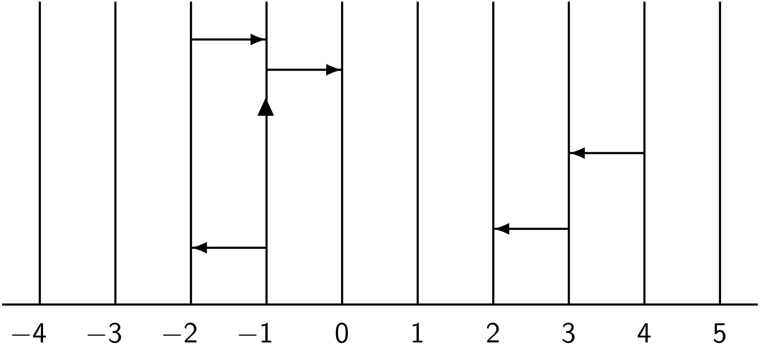
Let $\mathcal{N} = \{-1, 1\}$ be the set of (nearest) neighbours of 0. Then $y \in \mathbb{Z}$ is called a neighbour of $x \in \mathbb{Z}$ if $y - x \in \mathcal{N}$. The dynamics of the process $(\xi_t)_{t\geq 0}$ can be described as follows:

1. Particles die at rate 1;
2. At a free/empty position $x \in \mathbb{Z}$ a particle gets born at rate $\lambda \times$ the number of neighbouring sites that are occupied ($\lambda \in (0, \infty)$) /
each particle gives birth to a new particle at rate $\lambda \times |\mathcal{N}|$ and places the particle at a uniformly chosen neighboring site, if it is empty.

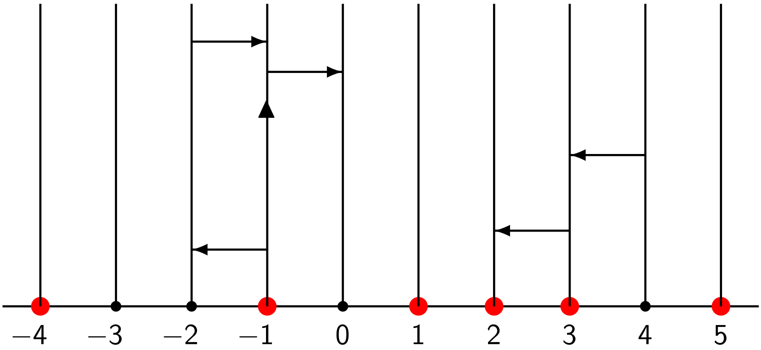
Remark: We can consider $(\xi_t)_{t\geq 0}$ as either

- a $\{0, 1\}^{\mathbb{Z}}$-valued process, where $\{x \in \mathbb{Z} : \xi_t(x) = 1\}$ denotes the set of occupied sites, or
- as a set-valued process, where $\xi_t \subset \mathbb{Z}$ denotes the set of occupied sites.

Graphical representation for $\mathcal{N} = \{-1, 1\}$



1. Drop triangles along each line at rate 1;
2. draw arrows from one fixed line to a neighboring (fixed) line at rate λ.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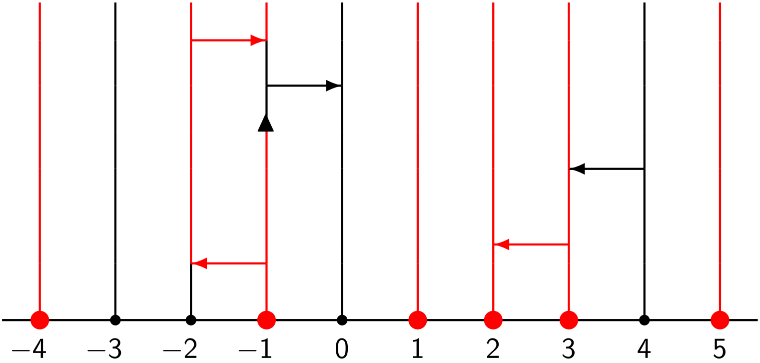
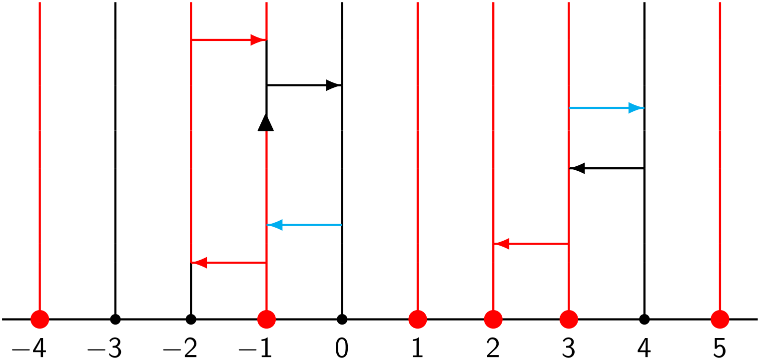


Useful properties (under the coupling based on the graphical representation): Let ξ_t^A denote the process starting with initial condition $\xi_0^A = A$. Then

- Set-monotonicity: If $A \subseteq B$, then $\xi_t^A \subseteq \xi_t^B$.
- Additivity: For all $A, B \subseteq \mathbb{Z}$, $\xi_t^A \cup \xi_t^B = \xi_t^{A \cup B}$.



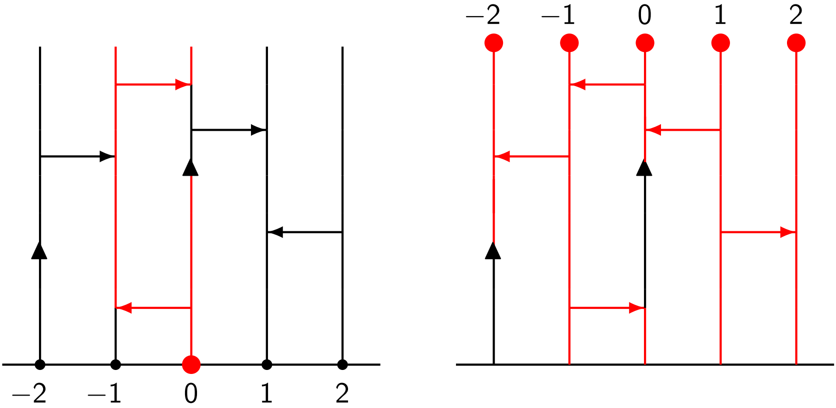
- λ-monotonicity: If $\lambda_1 \leq \lambda_2$, then $\xi_t(\lambda_1) \subseteq \xi_t(\lambda_2)$.

Idea: Independently add arrows at rate $\lambda_2 - \lambda_1 =: \Delta\lambda$.

- Self-duality:

$$\boxed{\mathbb{P}(\xi_t^A \cap B \neq \emptyset) = \mathbb{P}(A \cap \hat{\xi}_t^B \neq \emptyset) = \mathbb{P}(A \cap \xi_t^B \neq \emptyset)} \quad \text{for all } A, B \subseteq \mathbb{Z}, t \geq 0.$$

Example: $A = \{0\}$ (upward) and $B = \mathbb{Z}$ (downward; turn direction of arrows).



Conclusion: $\mathbb{P}(\xi_\cdot^{\{0\}}$ survives **globally**$) = \lim_{t\to\infty} \mathbb{P}(\xi_t^{\{0\}} \cap \mathbb{Z} \neq \emptyset)$

$= \lim_{t\to\infty} \mathbb{P}(\{0\} \cap \xi_t^{\mathbb{Z}} \neq \emptyset) = \mathbb{P}(\xi_\cdot^{\mathbb{Z}}$ survives **locally "at zero"**$)$.

Remark/Definition

Let $\xi_t^{\mathbb{Z}}$ be the process that starts in $\xi_0 = \mathbb{Z}$. Then $\xi_t^{\mathbb{Z}} \Rightarrow \xi_\infty^{\mathbb{Z}}$ for $t \to \infty$. $\xi_\infty^{\mathbb{Z}}$ has a translation invariant distribution μ (upper invariant measure) that satisfies

$$\mu(\cdot \cap B \neq \emptyset) = \mathbb{P}(\xi_\infty^{\mathbb{Z}} \cap B \neq \emptyset) = \mathbb{P}(\tau^B = \infty),$$

where $\tau^B := \inf\{t \geq 0; \xi_t^B \equiv \emptyset\}$ is the extinction time of the process starting with occupied sites B.

Set
$$\begin{aligned}\lambda_c &:= \inf\{\lambda \geq 0; \mu(\{\emptyset\}) < 1\} \\ &= \inf\{\lambda \geq 0; \mathbb{P}(\xi_\cdot^{\{0\}} \text{ survives } \textbf{globally}) > 0\}.\end{aligned}$$

Theorem (Harris' cvg.thm. for additive proc.s, [D95] (R. Durrett), p. 133)

For ξ_0 translation invariant with $\mathbb{P}(\xi_0 \equiv \emptyset) = 0$, $\boxed{\xi_t^{\xi_0} \Rightarrow \mu \text{ for } t \to \infty.}$

Theorem (Complete convergence theorem, [?] (T.M. Liggett), p. 284)

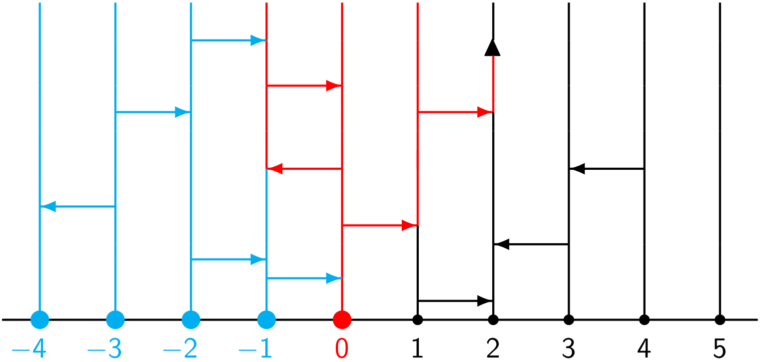
Let $\lambda > \lambda_c$. For any arbitrary initial distribution ξ_0,

$$\xi_t \Rightarrow \mathbb{P}(\tau^{\xi_0} < \infty) \cdot \delta_\emptyset + \mathbb{P}(\tau^{\xi_0} = \infty) \cdot \mu \quad \text{for } t \to \infty.$$

Edge speeds

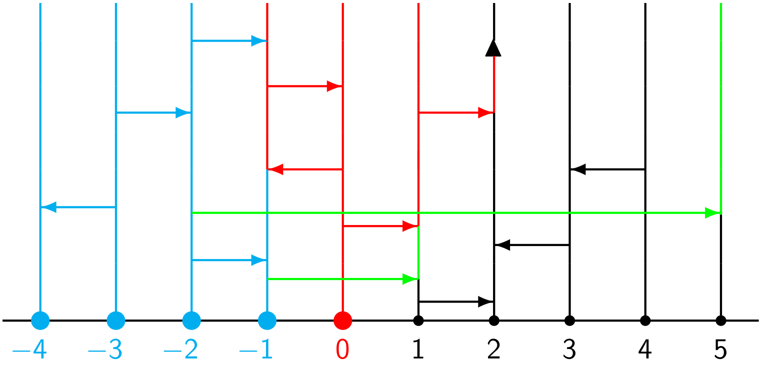
Definition (Edge processes)

Let $\ell_t^A := \inf\{x : x \in \xi_t^A\}$ *and* $r_t^A := \sup\{x : x \in \xi_t^A\}$ *for* $A \subset \mathbb{Z}$ *arbitrarily fixed.*



- $\ell_t^{\{0\}} = \ell_t^{[0,\infty)\cap\mathbb{Z}}$ and $\boxed{r_t^{\{0\}} = r_t^{(-\infty,0]\cap\mathbb{Z}} \quad \text{on } \{\tau^{\{0\}} > t\}.}$

A note on the long-range case



- $\ell_t^{\{0\}} \geq \ell_t^{[0,\infty)\cap\mathbb{Z}}$ and $r_t^{\{0\}} \leq r_t^{(-\infty,0]\cap\mathbb{Z}}$ on $\{\tau^{\{0\}} > t\}$.

In [D80, Theorem 1.4 and Section 4], Durrett shows for the *nearest-neigbor* contact process that

$$-\lim_{t\to\infty}\frac{l_t^{\{0\}}}{t} = \lim_{t\to\infty}\frac{r_t^{\{0\}}}{t} = \alpha \begin{cases} > 0, & \text{if } \tau^{\{0\}} = \infty, \\ < 0, & \text{if } \tau^{\{0\}} < \infty. \end{cases} \quad \text{a.s.}$$

Note: On $\{\tau^{\{0\}} = \infty\}$, $\boxed{\lim_{t\to\infty}\frac{r_t^{(-\infty,0]\cap\mathbb{Z}}}{t} = \lim_{t\to\infty}\frac{r_t^{\{0\}}}{t} = \alpha.}$

Useful property of $r_t^{(-\infty,0]\cap\mathbb{Z}}$:

Subadditivity in expectation: Let $\alpha_t := \mathbb{E}\big[r_t^{(-\infty,0]\cap\mathbb{Z}}\big]$. Then

$$\alpha_{t+u} < \alpha_t + \alpha_u, \quad t, u > 0$$

and

$$\lim_{T\to\infty}\frac{\alpha_T}{T} = \inf_{T>0}\frac{\alpha_T}{T} \text{ exists (and } = \alpha = \text{const.).}$$

Conclusion:

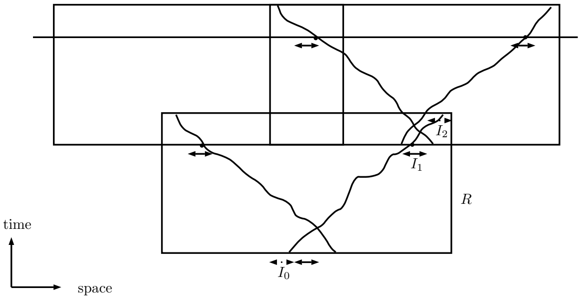
$$\boxed{\lambda_c = \sup\{\lambda \geq 0; \alpha(\lambda) < 0\} = \sup\{\lambda : \alpha(\lambda) \leq 0\}.} \tag{1}$$

Idea of the proof of complete convergence

(cf. [DG83] R. Durrett and D. Griffeath)

Let I_0 be a "big enough" interval with all sites occupied, so that the probability of survival is high.

A "successful path" from left to right starts in I_0, goes through I_1 and I_2, while never leaving the block R.



Now use comparison with 1-dependent oriented site-percolation with parameter close to 1.

Main part of the proof of (1)

Durrett proves in [D80, Section 4]: If $\lambda > \lambda_c$, then $\alpha(\lambda) \geq 0$ (easy) and for T big enough, $\left(\frac{\alpha_T}{T} = \frac{\mathbb{E}\left[r_T^{(-\infty,0]\cap\mathbb{Z}}\right]}{T} \overset{T\to\infty}{\to} \alpha \overset{!}{>} 0\right)$

$$\boxed{\alpha_T(\lambda+\delta) - \alpha_T(\lambda) \geq \delta T \quad \text{for all} \quad \delta \geq 0.}$$

Main ideas/steps of the proof.

1. For all B infinite subsets of $(-\infty, 0] \cap \mathbb{Z}$ and $t \geq 0$,
$$\mathbb{E}\left[r_t^{B\cup\{1\}} - r_t^B\right] \geq \mathbb{E}\left[r_t^{(-\infty,1]\cap\mathbb{Z}} - r_t^{(-\infty,0]\cap\mathbb{Z}}\right] = 1.$$
   - ▷ l.h.s.: use additivity, i.e. $\xi_t^{A\cup B} = \xi_t^A \cup \xi_t^B \Rightarrow r_t^{A\cup B} = \max\{r_t^A, r_t^B\}$;
   - ▷ r.h.s.: use translation-invariance.
2. Use λ-monotonicity for $\lambda_c < \lambda_1 < \lambda_2 = \lambda_1 + \Delta\lambda$ to show: There exists a.s. a finite (random) point in time τ s.t.
$$r_\tau^{(-\infty,0]\cap\mathbb{Z}}(\lambda_2) \geq r_\tau^{(-\infty,0]\cap\mathbb{Z}}(\lambda_1) + 1. \tag{2}$$
3. Split $[\lambda, \lambda+\delta]$ in $O(T)$ subintervals of length $\Delta\lambda = O(1/T)$. On each subinterval, (2) succeeds with probability $\Delta\lambda = O(1/T)$ on $[0,1]$ and with $O(1)$ on $[0,T]$ by a geometric-type series argument.

A summary

1. Let $\lambda > \lambda_c$.
2. Self-duality;
3. $\mathbb{P}(\xi_\infty^{\mathbb{Z}} \cap B \neq \emptyset) = \mathbb{P}(\tau^B = \infty)$.
4. Harris' convergence theorem.
5. Let $\alpha_t := \mathbb{E}\left[r_t^{(-\infty,0]\cap\mathbb{Z}}\right]$. Then
   - $\alpha_{t+u} < \alpha_t + \alpha_u, \quad t,u > 0$ and
   - $\lim_{T\to\infty} \alpha_T/T = \inf_{T>0} \alpha_T/T$ exists (and $=: \alpha =$ const.)
6. $\alpha_T(\lambda+\delta) - \alpha_T(\lambda) \geq \delta T$ for all $\delta \geq 0$.
   - ▷ $\mathbb{E}\left[r_t^{B\cup\{1\}} - r_t^B\right] \geq \mathbb{E}\left[r_t^{(-\infty,1]\cap\mathbb{Z}} - r_t^{(-\in\lambda_1 fty,0]\cap\mathbb{Z}}\right] = 1$,
   - ▷ $r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_2) \geq r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_1) + 1$.
7. $\lim_{T\to\infty} r_T^{(-\infty,0]\cap\mathbb{Z}}/T = \alpha$ a.s.
8. On $\{\tau^{\{0\}} = \infty\}$, $r_t^{(-\infty,0]\cap\mathbb{Z}} = r_t^{\{0\}}$.
9. Complete convergence theorem.

The KPP-equation with noise

The KPP equation with noise

We investigate solutions $u(t,x) = u_t(x) = u_t^{(u_0)}(x)$ to the stochastic partial differential equation (SPDE)

$$\begin{aligned} \partial_t u &= \partial_{xx} u + \theta u - u^2 + \sqrt{u} dW, \qquad t > 0, x \in \mathbb{R}, \theta > 0 \\ u(0,x) &= u_0(x) \geq 0. \end{aligned} \tag{3}$$

- $u \rightsquigarrow$ particle density,
- $\partial u_{xx} \rightsquigarrow$ particles move in space ($\mathbb{R}^1$) as independent Brownian motions,
- $\theta u \rightsquigarrow$ linear mass creation,
- $-u^2 \rightsquigarrow$ competition between particles "if they meet" / death due to overcrowding,
- $\sqrt{u} dW \rightsquigarrow$ standard deviation of particle branching (W a white noise).

Remark

Solutions to (3) *arise as limits of scaled* **long-range** *contact processes (cf. [MT95]* *(C. Mueller and R. Tribe, 1995))*.

Existing Results

Choose $\mathcal{C}_{tem}^{+}$ as type-stace, that is, the set of non-negative continuous functions with slower than exponential growth.

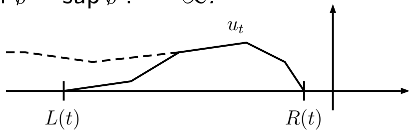
Theorem ([T96] (R. Tribe), Theorem 2.2)

(a) *For all $f \in \mathcal{C}_{tem}^{+}$ there exists a solution that starts in f.*

(b) *All solutions have the same law $\mathbb{P}_f$ and the strong Markov property holds. The map $f \mapsto \mathbb{P}_f$ is continuous.*

Definition

For $u_0 \in \mathcal{C}_{tem}^{+}$, let
$R(t) := \sup\{x \in \mathbb{R} : u_t(x) > 0\}$ *and*
$L(t) := \inf\{x \in \mathbb{R} : u_t(x) > 0\}$ *with* $\inf \emptyset = \sup \emptyset := -\infty$.



Remark

One can show: If $R(0) < \infty$, then $R(t) < \infty$ for all $t \geq 0$ a.s. A similar statement holds for $L(t)$.

The critical value θ_c

Definition

We say "u survives" if $\tau := \inf\{t \geq 0 : u_t \equiv 0\} = \infty$.

Theorem ([MT94] (C. Mueller and R. Tribe), Theorem 1)

Let $u(t,x)$ be a solution to

$$\frac{\partial u}{\partial t} = \Delta u + \theta u - u^2 + \sqrt{u}\dot{W}, \qquad t > 0, x \in \mathbb{R}, \theta > 0,$$
$$u(0,x) = u_0(x) \geq 0$$

with $u_0 \in \mathcal{C}_c^+$. Then there exists a constant $\theta_c > 0$, independent of u_0, such that:

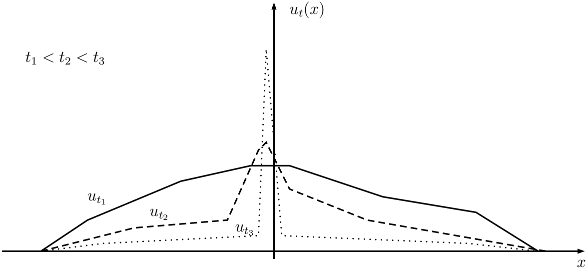
(a) *If $\theta < \theta_c$, then $\mathbb{P}_{u_0}(u \text{ survives}) = \mathbb{P}_{u_0}(\tau = \infty) = 0$.*

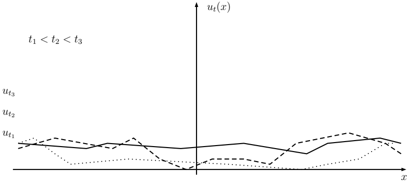
(b) *If $\theta > \theta_c$, then $\mathbb{P}_{u_0}(u \text{ survives}) = \mathbb{P}_{u_0}(\tau = \infty) > 0$.*

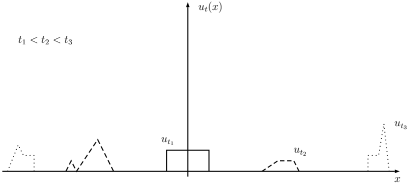
From now onwards, let $\theta > \theta_c$.

Longterm Behavior for arbitrary initial conditions?

Open Question: Complete convergence?







Analogue to Harris' convergence theorem for additive particle systems

Self-Duality

Let u, v be independent solutions to (3) with initial conditions $u_0, v_0 \in \mathcal{C}^+_{tem}$, then we have for all $0 \le s \le t$,

$$\mathbb{E}_{u_0}\left[e^{-2\langle u_t, v_0\rangle}\right] = \mathbb{E}_{u_0} \otimes \mathbb{E}_{v_0}\left[e^{-2\langle u_s, v_{t-s}\rangle}\right] = \mathbb{E}_{v_0}\left[e^{-2\langle u_0, v_t\rangle}\right],$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$.

Theorem ([HT04] (P. Horridge and R. Tribe), Theorem 1)

Let $\theta > \theta_c$. If $\nu \in \mathcal{P}(\mathcal{C}^+_{tem})$ only assigns mass to functions that are "uniformly distributed in space", then

$$\mathcal{L}(u_t^{(\nu)}) \Rightarrow \mu \text{ for } t \to \infty.$$

The limiting measure $\mu \in \mathcal{P}(\mathcal{C}^+_{tem})$ is unique and

- *translation invariant (in space), stationary, $\mu(f \not\equiv 0) = 1$*
- *and has as Laplace-functional*

$$\mathbb{E}\left[e^{-2\langle \mu, g\rangle}\right] := \int e^{-2\langle f, g\rangle}\mu(df) = \mathbb{P}_g(\tau < \infty), \qquad g \in \mathcal{C}^+_c.$$

A summary

1. Let $\lambda > \lambda_c$.
2. Self-duality;
3. $\mathbb{P}(\xi_\infty^{\mathbb{Z}} \cap B \neq \emptyset) = \mathbb{P}(\tau^B = \infty)$.
4. Harris' convergence theorem.
5. Let $\alpha_t := \mathbb{E}\left[r_t^{(-\infty,0]\cap\mathbb{Z}}\right]$. Then
   - $\alpha_{t+u} < \alpha_t + \alpha_u, \quad t, u > 0$ and
   - $\lim_{T\to\infty} \alpha_T / T = \inf_{T>0} \alpha_T / T$ exists (and $=: \alpha =$ const.)
6. $\alpha_T(\lambda + \delta) - \alpha_T(\lambda) \geq \delta T$ for all $\delta \geq 0$.
   - ▷ $\mathbb{E}\left[r_t^{B\cup\{1\}} - r_t^B\right] \geq \mathbb{E}\left[r_t^{(-\infty,1]\cap\mathbb{Z}} - r_t^{(-\infty,0]\cap\mathbb{Z}}\right] = 1$,
   - ▷ $r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_2) \geq r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_1) + 1$.
7. $\lim_{T\to\infty} r_T^{(-\infty,0]\cap\mathbb{Z}} / T = \alpha$ a.s.
8. On $\{\tau^{\{0\}} = \infty\}$, $r_t^{(-\infty,0]\cap\mathbb{Z}} = r_t^{\{0\}}$.
9. Complete convergence theorem.

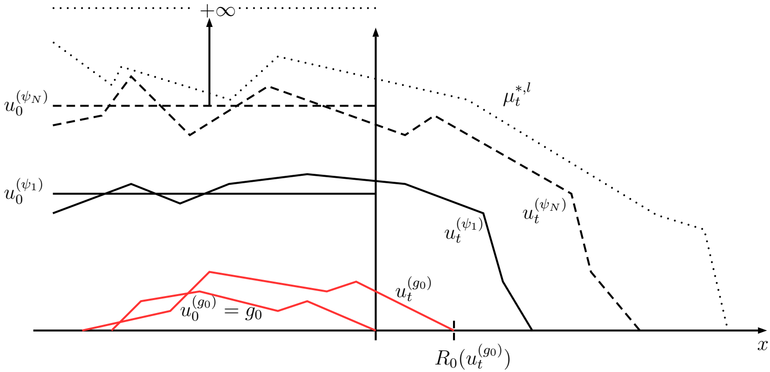
Analogue to $\xi_t^{(-\infty,0]\cap\mathbb{Z}}$ and $\xi_t^{\mathbb{Z}}$ (cf. [K17] (S. Kliem))

Let $\psi_n \uparrow "\infty" \cdot 1_{(-\infty,0)}$, then for $t > 0$,

$$\mathcal{L}(u_t^{(\psi_N)}) \Rightarrow \mathcal{L}(u_t^{*,l}) \in \mathcal{P}(\mathcal{C}_{tem}^{+}) \text{ for } N \to \infty.$$

Let $\psi_n \uparrow "\infty"$, then for $t > 0$,

$$\mathcal{L}(u_t^{(\psi_N)}) \Rightarrow \mathcal{L}(u_t^{*}) \in \mathcal{P}(\mathcal{C}_{tem}^{+}) \text{ for } N \to \infty.$$



Note: $\mathcal{L}(u_t^{*}) \Rightarrow \mu$ for $t \to \infty$ (μ, cf. Harris cvg. thm. [HT04]).

A summary

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   - $\alpha_{t+u} < \alpha_t + \alpha_u, \quad t, u > 0$ and
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   Becomes: Let $\alpha_t := \mathbb{E}\big[u_t^{*,l}\big]$. Then
   - $\alpha_{t+u} \leq \alpha_t + \alpha_u, \quad t, u > 0$ and
   - $\lim_{T\to\infty} \alpha_T / T = \inf_{T>0} \alpha_T / T$ exists (and $=: \alpha =$ const.)
6. $\alpha_T(\lambda + \delta) - \alpha_T(\lambda) \geq \delta T$ for all $\delta \geq 0$.
   - ▷ $\mathbb{E}\Big[r_t^{B\cup\{1\}} - r_t^B\Big] \geq \mathbb{E}\Big[r_t^{(-\infty,1]\cap\mathbb{Z}} - r_t^{(-\infty,0]\cap\mathbb{Z}}\Big] = 1$,
   - ▷ $r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_2) \geq r_T^{(-\infty,0]\cap\mathbb{Z}}(\lambda_2) + 1$.
7. $\lim_{T\to\infty} r_T^{(-\infty,0]\cap\mathbb{Z}} / T = \alpha$ a.s.
8. On $\{\tau^{\{0\}} = \infty\}$, $r_t^{(-\infty,0]\cap\mathbb{Z}} = r_t^{\{0\}}$.
9. Complete convergence theorem.

Travelling wave solutions

Let

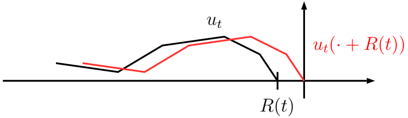
$$\boxed{\nu_T: \text{ the law of } \frac{1}{T}\int_0^T u_t(\cdot + R(s))ds} \text{ under } \mathbb{P}_{u_0},$$

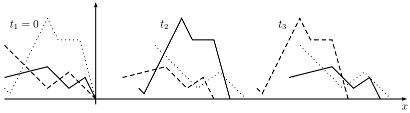
where $u_0 \in \mathcal{C}^+_{tem}$ arbitrary.

Definition

A *travelling wave solution* *to* (3), *is a solution with*

$$\begin{array}{ll} (i) & R(u(t)) \in (-\infty, \infty) \textit{ for all } t \geq 0, \\ (ii) & u(t, \cdot + R(u(t))) \textit{ is a (temporarily) stationary process.} \end{array}$$





1. For f_0 Heavyside initial data (and $R(t)$ replaced by $R_1(t) = \log(\int e^x u_t(x)dx)$), the sequence $(\nu_T)_{T\in\mathbb{N}}$ is tight. Every subsequential limit yields a travelling wave solution with "tip at zero" a.s. (cf. [T96]).
2. For $g_0 \in \mathcal{C}^+_c \setminus \{0\}$, the same holds true (conditional on survival) (cf. [K17]).

A summary

1. Let $\lambda > \lambda_c$.
2. Self-duality;
3. $\mathbb{P}(\xi_\infty^{\mathbb{Z}} \cap B \neq \emptyset) = \mathbb{P}(\tau^B = \infty)$.
4. Harris' convergence theorem.
5. **Becomes:** Let $\alpha_t := \mathbb{E}\left[u_t^{*,l}\right]$. Then
   - $\alpha_{t+u} \leq \alpha_t + \alpha_u, \quad t, u > 0$ and
   - $\lim_{T\to\infty} \alpha_T/T = \inf_{T>0} \alpha_T/T$ exists (and $=: \alpha =$ const.)
6. $\alpha_T(\lambda+\delta) - \alpha_T(\lambda) \geq \delta T$ for all $\delta \geq 0$.

   WiP: Let $\beta_T(\theta) := \beta_T := \frac{2}{T}\int_0^{T/2} \mathbb{E}\left[R(u^{*,l}_{T/2+s})\right] ds$. Then
   $$\beta_T(\theta_2) - \beta_T(\theta_1) \geq C(\theta_2 - \theta_1)T$$
   for all $\underline{\theta} \leq \theta_1 < \theta_2 \leq \overline{\theta}$ and T big enough.

   Conclusion: For all $\theta > \theta_c$, $\lim_{T\to\infty} \alpha_T/T > 0$.
7. $\lim_{T\to\infty} r_T^{(-\infty,0]\cap\mathbb{Z}}/T = \alpha$ a.s.
8. On $\{\tau^{\{0\}} = \infty\}$, $r_t^{(-\infty,0]\cap\mathbb{Z}} = r_t^{\{0\}}$.
9. Complete convergence theorem.

Open Questions

Note: θ_c is also the critical value for the existence of a nontrivial stationary distribution.

Open Questions:

1. Travelling wave speed $A = A(\nu) > 0$? Deterministic? Dependent on ν?
2. Does the limiting speed of an arbitrary solution to (3), $A(u_0) := \lim_{t\to} R(u_t)/t$, $u_0 \in \mathcal{C}^+_{tem}$ exist?
3. $A(\nu) = A(g_0), g_0 \in \mathcal{C}^+_c \backslash \{0\}$?
4. Suppose $A(\nu) > 0$ with positive probability for ν. Does that imply $A(g_0) > 0$ w.p.p. for g_0 with compact support?

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Thank You