# Weak universality of the parabolic Anderson model

Jörg Martin (HU Berlin)
joint work with Nicolas Perkowski

Spatial models in population genetics, Bath

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HUMBOLDT-UNIVERSITÄT ZU BERLIN    StoA RTG 1845    Berlin Mathematical School

Nonlinear PAM & weak universality

Recap: PAM on $\mathbb{R}^2$ & paraproducts

Proof of the weak universality

# Discrete PAM

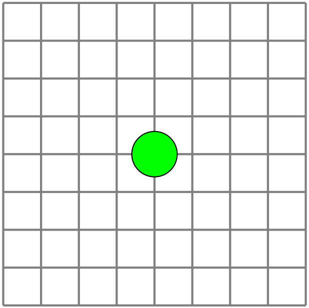

Figure: particle on $\mathbb{Z}^2$

- particle performs a random walk on $\mathbb{Z}^2$

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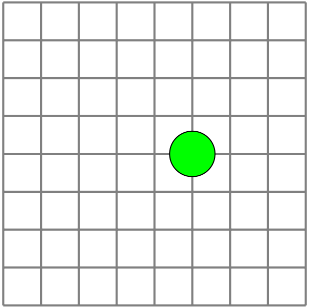

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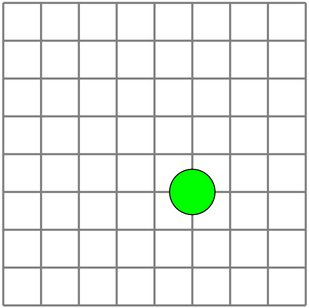

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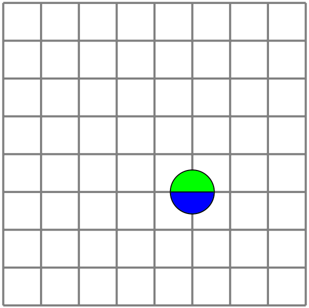

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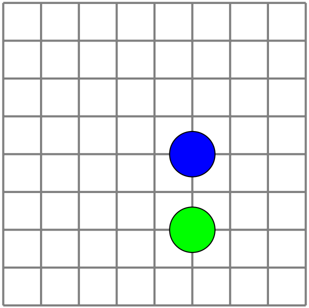

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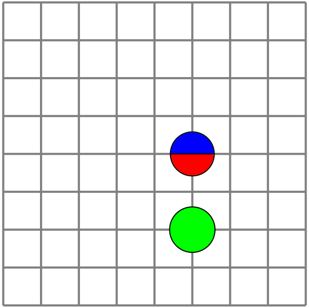

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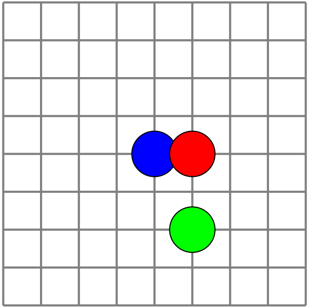

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- Intensively studied (e.g. Carmona, Molchanov, Gärtner, König,...)

# A second model

Population of some species $V : [0, T] \to [0, C]$, $C > 0$

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More general

$$\partial_t V = \lambda \cdot F(V),\ F(0) = 0.$$

# Nonlinear PAM

PAM & Growth Model → Nonlinear PAM

$$\partial_t v = \Delta_{\mathbb{Z}^2} v + \eta \cdot F(v)$$

- Individuals spread on $\mathbb{Z}^2$ ($\leftarrow \Delta_{\mathbb{Z}^2}$)

(If $F \geq 0$)

- $\eta(x) > 0$ → Population grows on $x$
- $\eta(x) \leq 0$ → Population decreases on $x$

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# PAM on $\mathbb{R}^2$

$$\partial_t u = \Delta u + F(u)(\xi - \infty)$$

solved by Gubinelli-Imkeller-Perkowski '15, using *paraproducts*, on $[0, T] \times \mathbb{T}^2$.
Linear equation ($F(u) = u$) was solved on $\mathbb{R}^2 \times [0, T]$ by Hairer-Labbé '15.

# Weak universality

We consider on $[0,T] \times \varepsilon\mathbb{Z}^2$

$$\partial_t u^\varepsilon = \Delta_{\varepsilon\mathbb{Z}^2} u^\varepsilon + F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon) \qquad (1)$$

where $F^\varepsilon = \varepsilon^{-2} F(\varepsilon^2 \cdot)$ and $c_\varepsilon \simeq |\log \varepsilon|$.

## Theorem (M.-Perkowski '17)

*If $u^\varepsilon$ solves (1) with $F \in C^2$, $F(0)=0$, $F'' \in L^\infty$, then there is an extension $\mathscr{E}^\varepsilon u^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$, $\mathscr{E}^\varepsilon u^\varepsilon|_{\varepsilon\mathbb{Z}^2} = u^\varepsilon$ such that for $\varepsilon \to 0$*

$$\mathscr{E}^\varepsilon u^\varepsilon \to u$$

*where $u$ solves on $[0,T] \times \mathbb{R}^2$*

$$\partial_t u = \Delta u + F'(0) u(\xi - \infty).$$

***"No saturation on large scales"***

# White noise & Hölder spaces

# White noise on $\mathbb{R}^2$

$\xi$ random measure (*not really*)

- $\forall \varphi \in L^2(\mathbb{R}^2) \qquad \xi(\varphi) := \int \varphi \, \mathrm{d}\xi \sim \mathcal{N}(0, \|\varphi\|_{L^2(\mathbb{R}^2)})$
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Recall: signed measures can be

- added
- multiplied with functions
- Fourier transformed

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but **NOT** defined: measure · measure!

(depends on "Hölder smoothness")

# Littlewood-Paley-blocks

For $f$ function or measure

$$f = \sum_{m\geq 0} \mathscr{F}^{-1}\Big(\mathbf{1}_{[2^m,2^{m+1})}(|\cdot|)\cdot\mathscr{F}f\Big) =: \sum_{m\geq 0}\Delta_m f$$

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Note: $\Delta_m f \in C^\infty$!

# Hölder spaces

For $\alpha \in \mathbb{R}$:

$$\mathscr{C}^\alpha(\mathbb{R}^2) := \left\{ f \,|\, \|f\|_{\mathscr{C}^\alpha(\mathbb{R}^2)} < \infty \right\},$$

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If $\alpha \in (0,1)$: functions with

$$\|f\|_{\mathscr{C}^\alpha(\mathbb{R}^2)} \approx \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

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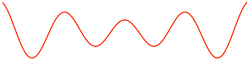

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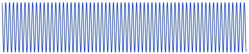

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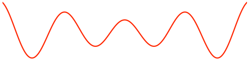

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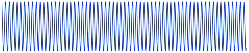

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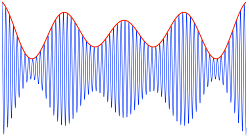

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$f \prec g, g \prec f$ are always well-defined,
$f \bullet g$ is only well-defined if $f \in \mathscr{C}^\alpha, g \in \mathscr{C}^\beta$ with $\alpha+\beta>0$.

# Properties of $\prec, \bullet$

- $(\partial_t - \Delta)(u \prec X) = u \prec (\partial_t - \Delta) X + \mathscr{R}$
- $u \bullet (v \prec w) = v \cdot (u \bullet w) + \mathscr{R}$
- $\xi \prec \mathscr{R} = \mathscr{R}$
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with a "smooth" term $\mathscr{R}$.

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Replace the *ill-defined* $\xi \bullet X$ by $\xi \diamond X = \lim_{n\to\infty} \xi^n \bullet X^n - \mathbb{E}[\xi^n \bullet X^n]$ ("Wick product")

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→ solution to linear PAM on $\mathbb{R}^2$

Nonlinear PAM & weak universality

Recap: PAM on $\mathbb{R}^2$ & paraproducts

Proof of the weak universality

Recall: We want to show for $u^\varepsilon$ in

$$\partial_t u^\varepsilon = \Delta_{\varepsilon\mathbb{Z}^2} u^\varepsilon + F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon)$$

with $F^\varepsilon = \varepsilon^{-2} F(\varepsilon^2 \cdot)$ converges (under some extension $\mathscr{E}^\varepsilon$) to

$$\partial_t u = \Delta u + F'(0) u(\xi - \infty)$$

# Previous methods

Previous method to combine discrete equations with "paraproduct approach" (Mourat-Weber '16, Gubinelli-Perkowski '17, Zhu-Zhu '15, Chouk-Gairing-Perkowski '17, Shen-Weber '16):

Find a $U^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$, $U^\varepsilon|_{\varepsilon\mathbb{Z}^2} = u^\varepsilon$ that satisfies similar equation on $\mathbb{R}^2$. Apply paraproduct methods on $U^\varepsilon$!

Random operators pop up → highly technical!

# New approach

- We introduce paraproducts on the lattice
- We develop a toolbox for discrete Wick calculus to handle renormalization (based on Caravenna-Sun-Zygouras '17)
- We derive uniform bound of $u^\varepsilon$ in discrete Hölder space
- We extend $u^\varepsilon$ to $\mathbb{R}^2$ "at the end" by some $\mathscr{E}^\varepsilon$, pass to the limit.

Our methods work for a wide class of SPDEs and shorten the proofs to quite some extend!

# Discrete paraproduct

$$f : \varepsilon\mathbb{Z}^2 \to \mathbb{C} \qquad \stackrel{\mathscr{F}}{\Longrightarrow} \qquad \mathscr{F} f : [-\pi\varepsilon, \pi\varepsilon]^2 \to \mathbb{C}$$

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$\Rightarrow$ All estimates carry over readily (in appropriate spaces)

# Discrete Hölder spaces

For $\alpha \in \mathbb{R}$:

$$\mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2) := \left\{ f \mid \|f\|_{\mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2)} < \infty \right\},$$

where

$$\|f\|_{\mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2)} := \sup_{-1\le j\le N^\varepsilon} 2^{j\alpha} \|\Delta_j f\|_{L^\infty(\varepsilon\mathbb{Z}^2)}$$

For example (with uniform bounds)

$$\xi^\varepsilon \in \mathscr{C}^{-1^-}(\varepsilon\mathbb{Z}^2) = \bigcap_{\alpha<-1} \mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2).$$

If $\alpha \in (0,1)$ (uniformly in $\varepsilon$)

$$\|f\|_{\mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2)} \approx \sup_{x\in\varepsilon\mathbb{Z}^2} |f(x)| + \sup_{x,y\in\varepsilon\mathbb{Z}^2, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

# A priori bounds for $(\partial_t - \Delta_{\varepsilon\mathbb{Z}^2})u^\varepsilon = F^\varepsilon(u^\varepsilon)\xi^\varepsilon$

Recall $F^\varepsilon = \varepsilon^{-2}F(\varepsilon^2\cdot)$, $F(0)=0$. We consider
$u^{\varepsilon,\sharp} := u^\varepsilon - F'(0)u^\varepsilon \prec X^\varepsilon, (\partial_t - \Delta_{\varepsilon\mathbb{Z}^2})X^\varepsilon = \xi^\varepsilon$

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&\overset{\text{Taylor!}}{=} F'(0)u^\varepsilon\xi^\varepsilon + o_\varepsilon(1) - (\partial_t - \Delta_{\varepsilon\mathbb{Z}^2})(F'(0)u^\varepsilon \prec X^\varepsilon)
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Then: basically same computations as above.

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Replace $\xi^\varepsilon \bullet X^\varepsilon$ by $\xi^\varepsilon \bullet X^\varepsilon - \overbrace{\mathbb{E}[\xi^\varepsilon \bullet X^\varepsilon]}^{c_\varepsilon \simeq |\log \varepsilon|}$

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Replace $\xi^\varepsilon \bullet X^\varepsilon$ by $\xi^\varepsilon \bullet X^\varepsilon - \overbrace{\mathbb{E}[\xi^\varepsilon \bullet X^\varepsilon]}^{c_\varepsilon \simeq |\log\varepsilon|}$
$\rightarrow$ Uniform bounds in discrete Hölder space

# Extension operator

Extend $u^\varepsilon : \varepsilon\mathbb{Z}^2 \to \mathbb{R}$ by $\mathscr{E}^\varepsilon u^\varepsilon : \mathbb{R}^2 \to \mathbb{R}, \mathscr{E}^\varepsilon|_{\varepsilon\mathbb{Z}^2} = u^\varepsilon$:

$$\mathscr{E}^\varepsilon u^\varepsilon = \mathscr{F}^{-1}_{\mathbb{R}^2}\left[\psi(\varepsilon\cdot)\cdot(\mathscr{F}_{\varepsilon\mathbb{Z}^2} u^\varepsilon)_{\text{ext}}\right]$$

$(\ldots)_{ext}$: periodic extension,
$\psi \in C_c^\infty(\mathbb{R}^2)$: $\sum_{k\in\mathbb{Z}^2}\psi(\cdot - k) = 1$.

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For $\alpha \in \mathbb{R}$ continuously (uniformly in $\varepsilon$)

$$\mathscr{E}^\varepsilon : \mathscr{C}^\alpha(\varepsilon\mathbb{Z}^2) \to \mathscr{C}^\alpha(\mathbb{R}^2)$$

Uniform bound of $u^\varepsilon$ in *discrete* Hölder space on $\varepsilon\mathbb{Z}^2$

$\downarrow$

Uniform bound of $\mathscr{E}^\varepsilon u^\varepsilon$ in Hölder space on $\mathbb{R}^2$

$\downarrow$

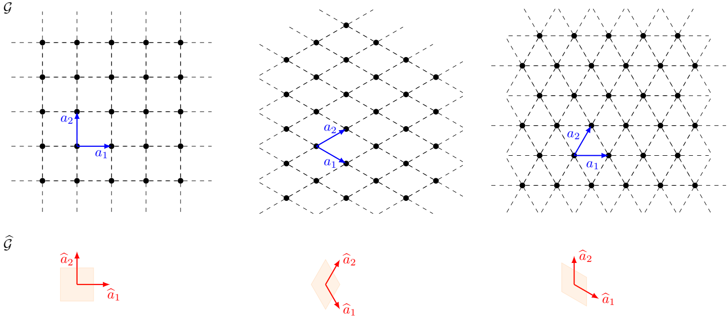
compact embedding: $\mathscr{E}^\varepsilon(u^\varepsilon) \longrightarrow u$

$\downarrow$

Convergence of equation

# Some generalizations

- Instead of $\varepsilon\mathbb{Z}^2$: $\varepsilon\mathscr{G}$ with crystal lattice $\mathscr{G} = a_1\mathbb{Z} + \ldots + a_d\mathbb{Z}$



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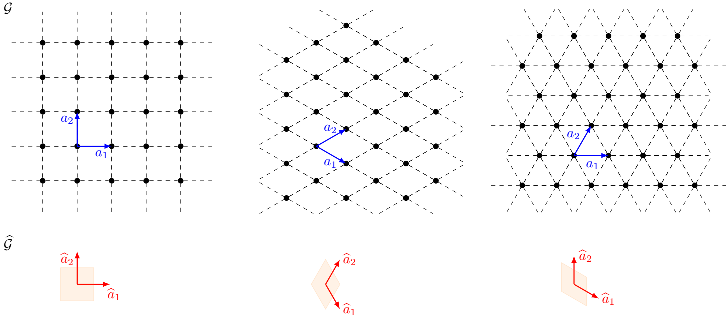


- Instead of $\Delta_{\varepsilon\mathbb{Z}^2}/\Delta_{\varepsilon\mathscr{G}}$: Generator of random walk with subexponential moments → constant-coefficient elliptic operator

Thank you for your attention!