On dual processes for additive and monotone interacting particle systems and applications

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Outline

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1. Duality concepts
2. A general construction of pathwise duals
3. Pathwise duality for monotone and additive processes

The classical duality concept

Let X and Y be two stochastic processes on some state spaces S and S'.

X and Y are **dual to each other with duality function** ψ if for $x \in S$ and $y \in S'$

$$\mathbb{E}^x[\psi(X_t, y)] = \mathbb{E}^y[\psi(x, Y_t)], \qquad t \geq 0.$$

(Roughly) equivalent:

- $G\psi = H\psi$ for G and H the generators of X and Y
- $s \mapsto \mathbb{E}[\psi(X_s, Y_{t-s})]$ is constant on $[0, t]$ with $t \geq 0$ when X and Y are independent.

Remark: **Sub/superduality** if equality is replaced by inequality.

Generalization of the concept: Pathwise duality

Y is a **(strong) pathwise dual** to X with duality function ψ if X and Y can be coupled such that

$$s \mapsto \psi(X_s, Y_{t-s})$$

is **almost surely constant** on $[0, t]$ with $t \geq 0$, and X_{s-} is independent of $Y_{t-s}, s \in [0, t]$.

Terminology, overview: Jansen and Kurt '14
More literature and examples later.

In **particle system/population genetics** context dual running backwards into the past as ancestral/genealogical process.

Outline

Random mapping representations of Markov processes

Let X be a continuous-time Markov chain with (finite) state space S and generator G. Then G can be written in the form of a **random mapping representation:**

Let $\mathcal{G} \subset \mathcal{F}(S,S) := \{m : S \to S\}$ and let $(r_m)_{m\in\mathcal{G}}$ be nonnegative constants.

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \quad , x \in S.$$

Note: This kind of representation is not unique.

The random mapping representation can be used for a Poissonian construction of the Markov process ($\rightarrow$ stochastic flow).

Poissonian construction of Markov processes

Let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$.
For $s \leq u$, set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s,u])$.
Define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\mathbf{X}_{s,t}(x) := m_n \circ \cdots \circ m_1(x) \text{ when}$$

$$\Delta_{s,t} := \{(m_1, t_1), \ldots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n.$$

Note that $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ for all $s \leq t \leq u$.

Poisson construction of Markov processes
Let X_0 be an S-valued r.v., independent of Δ. Setting for $s \in \mathbb{R}$,

$$X_t := \mathbf{X}_{s,s+t}(X_0), \qquad t \geq 0$$

defines a Markov process $X = (X_t)_{t \geq 0}$ with generator G.

Pathwise duality from the Poissonian construction

Let X and Y be continuous-time Markov chains with (finite) state spaces S and S' and generators

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big),$$

$$Hf(y) = \sum_{m\in\mathcal{G}} r_m\big(f(\hat{m}(y)) - f(y)\big).$$

Proposition (Pathwise duality)

Let $\psi : S \times S' \to \mathbb{R}$ be a function such that

$$(*) \qquad \psi(m(x), y) = \psi(x, \hat{m}(y)) \qquad x \in S,\ y \in S',\ m \in \mathcal{G}.$$

Then, X and Y are pathwise dual.

Proof: Use the Poissonian construction.

Construction of a pathwise dual

Goal:

- Construct in a general setting $\hat{m}$ and ψ such that $(*)$ holds:

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big).$$

General possibility Let $S' = \mathcal{P}(S)$, the set of all subsets of S, and

$$\hat{m}(A) = m^{-1}(A) := \{x \in S : m(x) \in A\}, \quad A \in \mathcal{P}(S).$$

Then equality holds in $(*)$ with respect to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}, \quad x \in S, A \in \mathcal{P}(S).$$

General duality function

$$\psi(x, A) := 1_{\{x \in A\}}, \quad x \in S, A \in \mathcal{P}(S).$$

"The dual with state space $\mathcal{P}(S)$ tracks the set of configurations that a particular (set of) configuration(s) may have emerged from."

This dual may be too unwieldy. $\Rightarrow$ Restrict the setting!

Find subspaces of $\mathcal{P}(S)$ that are invariant under the inverse image maps m^{-1} for all $m \in \mathcal{G}$.

Focus:

- Monotone and additive functions m on partially ordered sets.

Outline

Little excursion: Partially ordered sets

Let $(S, \leq)$ be a (finite) partially ordered set.

- For $A \subset S$ define $A^{\downarrow} := \{x \in S : x \leq y \text{ for some } y \in A\}$.
- $\mathcal{P}_{\text{dec}}(S)$ are the **decreasing** sets A with $A^{\downarrow} \subset A$.
- $\mathcal{P}_{!\text{dec}}(S)$ is a **principal ideal** if it consists of A with

$$A = \{z\}^{\downarrow} \text{ for some } z \in S.$$

Define analogously $A^{\uparrow}$,
increasing sets $\mathcal{P}_{\text{inc}}(S)$ and **principle filters** $\mathcal{P}_{!\text{inc}}(S)$.

Little excursion: Partially ordered sets

- In a join-semilattice $\mathcal{P}_{!\mathrm{inc}}(S)$ is closed under finite intersections and the **supremum** is well defined via

$$\{x \vee y\}^{\uparrow} := \{x\}^{\uparrow} \cap \{y\}^{\uparrow}$$

- $x \vee y$ is the minimal element such that

$$x \leq x \vee y \quad \text{and} \quad y \leq x \vee y.$$

- For S finite or bounded join-semilattice we have $\emptyset \neq A \subset \mathcal{P}_{!\mathrm{dec}}(S) \Leftrightarrow$ $A \subset \mathcal{P}_{\mathrm{dec}}(S)$ and $x, y \in A$ implies $x \vee y \in A$.

Example:

In the context of interacting particle systems choose for example

- $S = \mathcal{P}(\Lambda)(\cong \{0,1\}^{\Lambda})$ with partial order $\subset$.
- Here, $\vee$ corresponds to $\cup$.

Little excursion: Monotone and additive functions

- A function m is **monotone** if

$$x \leq y \text{ implies } m(x) \leq m(y), \quad x, y \in S.$$

- A function m is **additive** on a join-semilattice with minimal element 0 if

$$m(x \vee y) = m(x) \vee m(y), \quad x, y \in S$$

as well as $m(0) = 0$.

Remark:

- Additive functions are monotone.

Invariant subspaces for monotone and additive functions

Proposition (Monotone functions)

Equivalent:

- m is monotone.
- m^{-1} maps $\mathcal{P}_{\rm dec}(S)$ into itself (invariant subspace!).
- m^{-1} maps $\mathcal{P}_{\rm inc}(S)$ into itself (invariant subspace!).

Proposition (Additive functions)

Equivalent (on a finite join-semilattice with minimal element):

- m is additive.
- m^{-1} maps $\mathcal{P}_{\rm !dec}(S)$ into itself (invariant subspace!).

- $m^{-1}(A) \in \mathcal{P}_{\rm dec}(S)$ for $A \in \mathcal{P}_{\rm !dec}(S)$ (additive functions monotone)
- $x, y \in m^{-1}(A) \Rightarrow x \vee y \in m^{-1}(A)$
  since $m(x \vee y) = m(x) \vee m(y)$ and $m(x) \vee m(y) \in A$.

Monotonically and additively representable processes

If a Markov process X has random mapping representation

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m(f(m(x)) - f(x)) \quad , x \in S$$

where

- $\mathcal{G}$ contains only monotone functions then we call X **monotonically representable.**
- $\mathcal{G}$ contains only additive functions then we call X **additively representable.**

Pathwise duality for additively representable processes

S' is a *dual* of S if there is a bijection $S \ni x \mapsto x' \in S'$ $(x'' = x)$ with

$$x \leq y \quad \Leftrightarrow \quad x' \geq y'.$$

Examples:

- **1** $S' := S$ equipped with the reversed order and $x' = x$.
- **2** For $S \subset \mathcal{P}(\Lambda)$ equipped with $\subset$ take for $x' := \Lambda \backslash x = x^C$, the complement of x, and $S' := \{x' : x \in S\}$.

Now consider for $x \in S,\ y \in S'$

$$\psi(x,y) = 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}}$$

- **1** $\psi(x,y) = 1_{\{x \leq y'\}} = 1_{\{x \leq y\}}$
  Siegmund's duality on a totally ordered space S
- **2** $\psi(x,y) = 1_{\{x \subset \Lambda \backslash y\}} = 1_{\{x \cap y = \emptyset\}}$
  Additive interacting particle systems

Pathwise duality for additively representable processes

Lemma (Duals to additive maps)

For additive $m : S \to S$ there exists (a unique) $m' : S' \to S'$ with

$$(*) \qquad 1_{\{m(x) \leq y'\}} = 1_{\{x \leq (m'(y))'\}}, \qquad x \in S, y \in S'.$$

Proof

- By additivity m^{-1} maps sets of the form

$$A = \{y'\}^{\downarrow} = \{x \in S : x \leq y'\}, \quad y \in S'$$

into sets of this form.

- Thus, there exists an element $z \in S$ such that

$$m^{-1}\big(\{x \in S : x \leq y'\}\big) = \{x \in S : x \leq z\}$$

Set $m'(y) = z', y \in S'$

$$\Leftrightarrow m(x) \leq y' \quad \text{if and only if} \quad x \leq (m'(y))'$$

Pathwise duality for additively representable processes

Theorem (Additive systems duality)

Let S be a finite lattice and let X be a Markov process in S whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big), \qquad x \in S,$$

where all maps $m \in \mathcal{G}$ are additive **(additively representable)**. *Then the Markov process Y in S' with generator*

$$Hf(y) := \sum_{m \in \mathcal{G}} r_m \big(f(m'(y)) - f(y) \big), \qquad y \in S'$$

is pathwise dual to X with respect to the duality function

$$\psi(x, y) = 1_{\{x \leq y'\}}, \quad x \in S, y \in S'.$$

Percolation structure for additively representable processes

Equip $S := \mathcal{P}(\Lambda)$ with $\subset$ and let m be an additive map $S \to S$. Define $M \subset \Lambda \times \Lambda$ via

$$m(x) = \{j \in \Lambda : (i,j) \in M \text{ for some } i \in x\} \qquad x \in S.$$

Vice versa, any such $M \subset \Lambda \times \Lambda$ corresponds to an additive map m.

Percolation structure for additively representable processes

Let $S' = S$ and $x' = x^C$. Then we have an additive $m' : S \to S$ dual to m with the duality function

$$\psi(x, y) = 1_{\{x \subset \Lambda \setminus y\}} = 1_{\{x \cap y = \emptyset\}}, \quad x, y \in S.$$

The $M' \subset \Lambda \times \Lambda$ corresponding to m' is given by

$$M' = \{(j, i) : (i, j) \in M\}.$$

Percolation structure for additively representable processes

Percolation representation

Plot space-time $\Lambda \times \mathbb{R}$ with time upwards.
At rate r_m we consider the M associated to m and

- draw an arrow from (i, t) to (j, t) $(i \neq j)$ whenever $(i, j) \in M$
- place a "blocking symbol" at (i, t) whenever $(i, i) \notin M$

"Open paths" $\rightsquigarrow$ travel upwards along arrows and avoid blocking symbols. Then

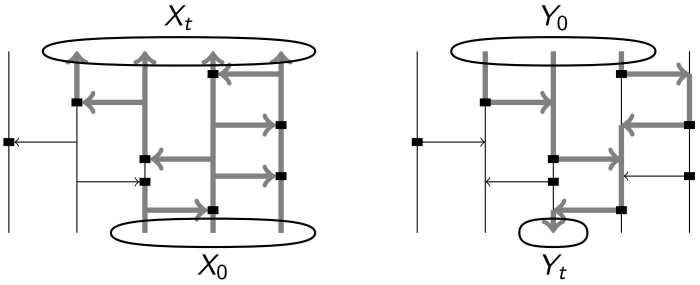
$$\mathbf{X}_{s,u}(x) = \{j \in \Lambda : (i, s) \rightsquigarrow (j, u) \text{ for some } i \in x\},$$

and the dual process is obtained via open paths using the reversed arrows (in reversed time).

Percolation structure for additively representable processes

Voter model

$S = \{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$.



Percolation structure for additively representable processes

Extensions

The above percolation structure statements also apply if

- Λ is a partially ordered set and $S = \mathcal{P}_{\mathrm{dec}}(\Lambda)$.
- S is a **distributive lattice** with

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad x, y, z \in S.$$

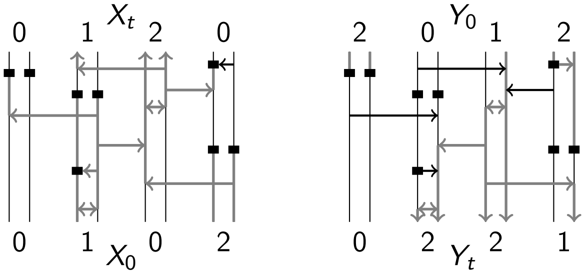
One can show that $S \cong \mathcal{P}_{\mathrm{dec}}(\Lambda)$ for a partially ordered set Λ by Birkhoff's representation theorem.

In this case for $i, j, i', j' \in \Lambda$

$$\text{(i)} \quad (i, j) \in M \text{ and } i \leq i' \text{ implies } (i', j) \in M,$$
$$\text{(ii)} \quad (i, j) \in M \text{ and } j \geq j' \text{ implies } (i, j') \in M.$$

Percolation structure for additively representable processes

Two stage contact process (Krone '99)
$S = \{0,1,2\}^\Lambda$ "1" younger individual "2" older individual.
Older individuals give birth to younger individuals who "grow up" and possibly die at a higher rate than older individuals.
$S \cong \mathcal{P}_{\rm dec}(\Lambda \times \{0,1\})$.



Pathwise duality for monotonically representable processes

Now consider the duality function

$$\phi(x,B) := 1_{\{x \le y' \text{ for some } y \in B\}}, \qquad x \in S,\ B \in \mathcal{P}(S').$$

Lemma (Duals to monotone maps)

For monotone $m : S \to S$ there exist $m^* : \mathcal{P}(S') \to \mathcal{P}(S')$ with

$$(*) \qquad 1_{\{m(x) \le y' \text{ for some } y \in B\}} = 1_{\{x \le y' \text{ for some } y \in m^*(B)\}}.$$

Proof idea

- By monotonicity m^{-1} maps decreasing sets of the form

  $$A = \{B'\}^{\downarrow} = \{x \in S : x \le y' \text{ for some } y \in B\}, \quad B \in \mathcal{P}(S')$$

  into sets of this form.
- Construct appropriate m^* : $m^*(B)' := \bigcup_{x \in B} (m^{-1}(\{x'\}^{\downarrow}))_{\max}$

Pathwise duality for monotonically representable processes

Theorem (Monotone systems duality)

Let S be a finite partially ordered set and let X be a Markov process in S whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big) \quad x \in S,$$

*where all maps $m \in \mathcal{G}$ are monotone (**monotonically rep.**). Then the $\mathcal{P}(S')$-valued Markov process Y^* with generator*

$$H_* f(B) = \sum_{m \in \mathcal{G}} r_m \big(f(m^*(B)) - f(B) \big), \quad B \in \mathcal{P}(S')$$

is pathwise dual to X with respect to the duality function ϕ.

Pathwise duality for cooperative branching coalescent

State space

- $\Lambda = (V, E)$ be a countable, connected, vertex transitive (degree D), locally finite graph with vertex set V and set of (undirected) edges E
- $S = \mathcal{P}(\Lambda) \cong \{0, 1\}^{\Lambda}$

Examples:

- $\Lambda = \mathbb{Z}^d$ with nearest-neighbor edges ($D = 2d$)
- $\Lambda = K_N$ complete graph ($D = N - 1$)
- $\Lambda = \mathbb{T}_d$ a regular tree ($D = d + 1$)

Pathwise duality for cooperative branching coalescent

Continuous-time Markov process $X = (X_t)_{t\geq 0}$ on $\{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$

- **Pairs of particles produce a new particle:** $110 \to 111$

  map $\texttt{coop}_{ijk}$ for $\langle i,j\rangle, \langle j,k\rangle \in E$ at rate $\beta\frac{1}{D(D-1)}$

  for particles at sites i and j producing a particle at site k
- **Symmetric random walk with coalescence:** $10, 11 \to 01$

  map $\texttt{rw}_{ij}$ for $\langle i,j\rangle \in E$ at rate $\gamma\frac{1}{D}$

  particle moving from i to j merging with any particle present
- **Spontaneous death of particles:** $1 \to 0$

  map $\texttt{death}_i$ at rate δ

  particle at site i disappears

Remark: One may also include voter $\texttt{vot}_{ij}$: $01 \to 11, 10 \to 00$ and exclusion dynamics $\texttt{exc}_{ij}$: $10 \to 01, 01 \to 10$

Pathwise duality for cooperative branching coalescent

Examples considered:

- $\Lambda = \mathbb{Z}$ without spontaneous death:
  Sturm, Swart '15
- $\Lambda = K_N$ complete graph without random walk
  (also $\Lambda = \mathbb{T}_d, \mathbb{Z}^d$):
  Mach, Sturm, Swart, in progress '17

Pathwise duality for cooperative branching coalescent

All maps m are monotone, all but cooperative branching are additive. Let $S' = S$ and $x' = x^C$. Then the duality function is

$$\phi(x, B) = 1_{\{x \subset y^c \text{ for some } y \in B\}} = 1_{\{x \cap y = \emptyset \text{ for some } y \in B\}}$$

for $x \in S, B \in \mathcal{P}(S)$.

- For the additive functions m there are dual functions m' with

$$m(x) \cap y = \emptyset \Leftrightarrow x \cap m'(y) = \emptyset$$

and we set $m^*(B) = \{m'(x) : x \in B\}$. We have

$$\texttt{rw}'_{ij} = \texttt{vot}_{ij}, \quad \texttt{death}'_i = \texttt{death}_i, \quad \texttt{vot}'_{ij} = \texttt{rw}_{ij}, \quad \texttt{exc}'_{ij} = \texttt{exc}_{ij}$$

Pathwise duality for cooperative branching coalescent

- For the cooperative branching map we have

$$\mathrm{coop}^*_{ijk}(B) = b^{(1)}_{ijk}(B) \cup b^{(2)}_{ijk}(B)$$

with the definition (restricted to sites ijk)

$$b^{(1)}: \ 001 \to 011, \quad b^{(2)}: \ 001 \to 101$$

since

$$\big(coop^{-1}(\{x\}^{\downarrow})\big)_{\max} = \begin{cases} \{100, 010\} & \text{if } x = 110, \\ \{x\} & \text{otherwise.} \end{cases}$$

and $x' := x^C$.

Pathwise duality for cooperative branching coalescent

Sturm, Swart '15

$\Lambda = \mathbb{Z}$ without spontaneous death

- **Application of a version of this dual:**
  Decay rates of the survival probability and the density in the subcritical regime is order $t^{-1/2}$
- **Additional results regarding phase transitions**

$$\beta_{\text{surv}} := \inf\{\beta > 0 : \text{ the process survives}\},$$
$$\beta_{\text{upp}} := \inf\{\beta > 0 : \text{ the upper invariant law is nontrivial}\}.$$

We have $1 \leq \beta_{\text{surv}}, \beta_{\text{upp}} < \infty$.

Pathwise duality for cooperative branching coalescent

Let $S' = S$ with reversed order and consider the duality function

$$\tilde{\phi}(x, B) = 1_{\{x \geq y \text{ for some } y \in B\}}, \qquad x \in \{0,1\}^\Lambda, B \in \mathcal{P}(S).$$

By considering $\left(m^{-1}(\{x\}^\uparrow)\right)_{\min}$ obtain the **dual maps**

- **Double branching map** $\texttt{coop}^*_{ijk}(B) = B \cup \texttt{dbran}_{ijk}(B)$ with the map $\texttt{dbran}_{ijk}: \quad 001, 011, 101, 111 \to 110$
- **Random walk map** $\texttt{rw}^*_{ij}(B) := \{y \in B : y(i) = 0\} \cup \texttt{e}_{ij}(B)$ with the map $\texttt{e}_{ij}: \quad 01 \to 10$
- **Death map** $\texttt{death}^*_i(B) := \{y \in B : y(i) = 0\}$.

Pathwise duality for cooperative branching coalescent

Mach, Sturm, Swart '17+

Model with cooperative branching and spontaneous death.

- Application of this dual to characterize the behavior of the process and its dual on K_N for $N \to \infty$ (mean field model).
- $\beta_{upp} < \beta_{surv}$ on $\mathbb{T}_d$ with $d \geq 9$.
- $\beta_{upp} \leq \beta_{surv}$ on $\mathbb{Z}_d$ (conjecture $\beta_{upp} = \beta_{surv}$).

Pathwise duality for monotone and additive processes

Conclusion:

- General framework for obtaining duals, in particular for (monotone, additive) spatial interacting particle systems.
- Some duals may be interpreted as potential ancestors/genealogies.